Escape from noisy intermittent repellers

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Intermittent or marginally stable repellers are commonly associated with a power law decay in the survival fraction. We demonstrate how the presence of weak additive noise alters the spectrum of the Perron-Frobenius operator significantly, giving rise to exponential decays even in systems that are otherwise regular. Implications for ballistic transport in marginally stable miscrostructures are also discussed.

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There exist a variety of physical situations where one deals with the escape of trajectories from repellers. For instance, nuclear physicists are interested in the escape of particles along fission channels. Similarly, in the study of transport coefficients in two-dimensional ballistic conductors, one has to worry about the trapping time and its relationship with the geometry of the system [1]. By and large, it is now accepted that hyperbolic (chaotic) dynamics leads to an exponential decay in the number of trapped particles while intermittency or marginal stability results in power law decays. Thus, a rectangular billiard table with a hole in the wall shows a power law decrease in the *survival fraction* while a (hyperbolic) enclosure created by the intersection of three disks shows an exponential decay in the number of trapped particles. The reason for this difference is intuitively clear from the following argument: Consider that there are N particles distributed uniformly in (Birkhoff) phase space [2] and there is a hole along the wall at q_0 of extent Δq_0. The (average) fraction of particles that escape at each bounce is identical for a chaotic system and proportional to Δq_0 since the particles remain uniformly distributed with time. Thus the survival fraction decays exponentially. In a marginally stable system, however, an initial uniform distribution does not remain uniform at each bounce since individual particles tend to stick around stable islands. A heuristic derivation of the power law decay can be found in [3], and we merely remark here that the decay exponent is often difficult to determine analytically and that an interesting advancement in this direction has been achieved recently by Dalhqvist [4].

We are interested here in a situation where marginal stability or intermittency is accompanied by weak additive noise. Such a situation can arise, for instance, in an imperfectly fabricated ballistic conductor in the shape of a triangle or stadium where reflection is no longer specular but has additive noise. Thus

$$\begin{aligned} q_{n+1} &= f_1(q_n, p_n), \\ p_{n+1} &= f_2(q_n, p_n) + \xi_n, \end{aligned} \tag{1}$$

where q,p are the Birkhoff coordinates [2], f_1, f_2 are the bounce maps, and ξ_n is a random variable with $\langle \xi_n \rangle = 0$ having a normalized distribution $g(\xi)$ (normally taken to be a Gaussian with zero mean). The question that we pose is: *does one expect to find a power law decay in such a situation*? The answer, we believe, is interesting and can significantly alter the way people look at signatures of low-dimensional chaos in various experimental situations where noise is inevitable and often desirable. First, however, we shall consider a one-dimensional intermittent map and study its spectrum in the presence of weak noise.

A trajectory in the presence of additive noise is generated by the iteration

$$x_{n+1} = f(x_n) + \xi_n, \tag{2}$$

where $f(x)$ is a map, ξ_n is a random variable as described above, and $x_0 \in [a,b]$. An initial density of trajectories $\phi(x)$ evolves according to the Perron-Frobenius equation [5],

$$(\mathcal{L}_0 \circ \phi)(x) = \int dy\, \delta(x - f(y))\phi(y), \tag{3}$$

in the unperturbed case. Thus, the eigenvalues and eigenfunctions of $\mathcal{L}_0$ determine the escape rate in an open system. More specifically, assuming that the spectrum is discrete, an initial density $\phi(x)$ can be expanded as

$$\phi(x) = \sum_\alpha c_\alpha \varphi_\alpha(x), \tag{4}$$

so that the fraction of particles that survive n iterates of the map is

$$\begin{aligned} \Gamma(n) &= \frac{\int_a^b dx (\mathcal{L}_0^n \circ \phi)(x)}{\int_a^b dx\, \phi(x)} = \sum_\alpha \Lambda_\alpha^n c_\alpha \frac{\int_a^b dx\, \varphi_\alpha(x)}{\int_a^b dx\, \phi(x)} \\ &\sim \Lambda_0^n = e^{-n \ln(1/\Lambda_0)} \quad \text{as} \quad n \to \infty. \end{aligned} \tag{5}$$

In the above, $\{\Lambda_\alpha\}$ are the eigenvalues corresponding to the eigenfunctions $\{\varphi_\alpha(x)\}$ and Λ_0 is the *leading* eigenvalue with the largest real part. The discreteness assumption, however, holds only when the dynamics is hyperbolic. In the presence of marginally stable cycles, the spectrum has a continuous part leading to a power law decay of correlations (in closed systems) or survival fraction (in open systems).

The presence of additive noise results in a modified kernel whose formal expression is well known [6]:

$$(\mathcal{L}\circ\phi)(x)=\int dy\,\mathcal{L}(x,y)\,\phi(y), \tag{6}$$

where

$$\mathcal{L}(x,y)=\delta_\sigma(x-f(y)) \tag{7}$$

$$\delta_\sigma(x)=\int \delta(x-\xi)g(\xi)d\xi=g(x). \tag{8}$$

As before, if the spectrum of $\mathcal{L}$ is discrete, the decay is exponential and the leading eigenvalue determines the asymptotic decay rate.

Note that we have so far steered clear of spillover effects due to noise [7]. The most commonly adopted technique is the use of *periodic boundary conditions*, which avoids spillover altogether. Alternately, one can work in the infinite domain $(-\infty,\infty)$ so that natural boundary conditions may be employed. Yet another approach is to tailor the noise distribution so that the probability of the dynamical variable escaping from the interval is zero. We shall, in this paper, have occasion to use the second and third approaches depending on the problem, and it must be noted that there are other approaches to the spillover problem that may be more realistic in a given situation. Note that the spectrum and eigenfunctions of $\mathcal{L}$ can be sensitive to the choice of boundary conditions.

With this background, we now introduce the intermittent map [4,8]

$$f(x)=\begin{cases} x(1+x^2), & x<0\\ x[1+p(2x)^s], & 0\leqslant x<1/2\\ 2x-1, & x>1/2,\end{cases} \tag{9}$$

where $s>0$ and $p>1$. The particle is considered to escape if $x_{n+1}<0$ or $x_{n+1}>1$. The map is defined in the infinite domain so that natural boundary conditions apply on the density. The intermittency here is due to the fact that $f'(0)=1$ so that the fixed point $x=0$ is marginally or neutrally stable.

For an initial *uniform* distribution of particles in [0,1], the fraction that survives one iterate is clearly the sum of the two intervals I_L and I_R for which $0\leqslant f(x)\leqslant 1$. Similarly, the fraction that survives two iterates is the sum of the four intervals $I_{LL},I_{LR},I_{RL},I_{RR}$ for which $0\leqslant f^2(x)\leqslant 1$. Generalization in this case (of binary symbolic dynamics) is simple: the fraction that survives n iterates is the sum of the 2^n intervals for each of which $0\leqslant f^n(x)\leqslant 1$. Each of these intervals contains a periodic point and the larger its (in)stability, the smaller the size of the interval. Thus [5]

$$I_q^{\{n\}}=\frac{a_q}{\Lambda_q}, \tag{10}$$

where q is a symbol sequence of length n consisting of L and R, which denotes the order in which the left and right branches (with respect to $x=1/2$) of the map are visited, a_q is a constant and $\Lambda_q=(d/dx)f^n(x)|_{x\in I_q}$ is the stability of the periodic point. The survival fraction can thus be expressed as [4,5]

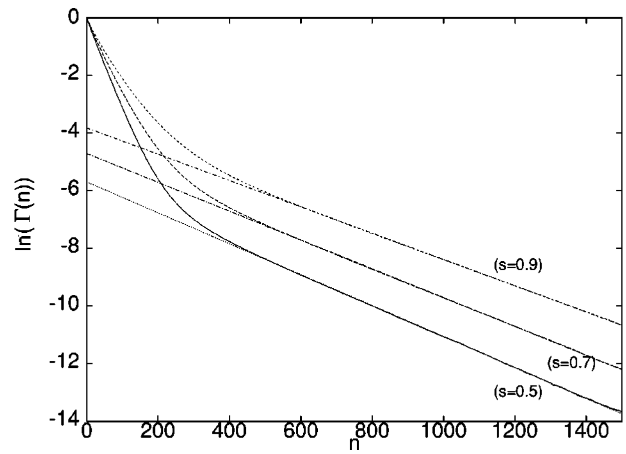

FIG. 1. The survival fraction $\Gamma(n)$ for $s=0.5$, 0.7, and 0.9 with $\sigma=0.002$. The straight lines (dashed) are the best fit exponential decays. The initial distribution in each case consists of 10^{10} points.

$$\Gamma(n)=\sum_q^{\{n\}} I_q=\sum_q^{\{n\}}\frac{a_q}{\Lambda_q}\sim\sum_p\sum_{r=1}^{\infty}\frac{n_p\delta_{n,rn_p}}{|\Lambda_p|^r}=\mathcal{Z}_n,$$
$$\mathcal{Z}_n=\frac{1}{2\pi i}\int_\gamma z^{-n}\left(\frac{d}{dz}\ln\zeta^{-1}(z)\right)dz, \tag{11}$$

where $\zeta^{-1}(z)=\Pi_p(1-z^{n_p}/|\Lambda_p|)$ is the dynamical zeta function and γ is a (small) negatively oriented contour around the origin. Dahlqvist [4] has recently shown that, in the *noiseless* case, $\zeta^{-1}(z)$ has a singularity of the type $(1-z)^{1/s}$. It then follows [4] from Eq. (11) that the survival fraction $\Gamma(n)\sim 1/n^{1/s}$ for an initial *uniform* distribution of particles.

The ζ function is also (approximately) related to the eigenvalues of $\mathcal{L}$ through the relation

$$\mathrm{Tr}\,\mathcal{L}^n=\sum_\alpha \Lambda_\alpha^n=\sum_p\sum_{r=1}^{\infty}\frac{n_p\delta_{n,rn_p}}{|1-\Lambda_p|^r}\simeq\mathcal{Z}_n. \tag{12}$$

When the ζ function is analytic, its zeros $\{z_k\}$ are isolated and related to Λ_α as $\Lambda_\alpha=1/z_k$. On the other hand, when the system is intermittent and $\zeta^{-1}(z)$ displays a branch cut, the spectrum of $\mathcal{L}$ no longer remains discrete. Thus intermittency leads to an asymptotic power law decay. For $s=0.7,p=1.2$, the initial decay is exponential, however, and this is ascribed to a pair of complex conjugate roots [4]. The power law behavior emerges only after 600 iterations of the map.

We now consider the map [Eq. (9)] with weak additive Gaussian noise

$$g(\xi)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\xi^2/2\sigma^2} \tag{13}$$

for $s=0.9$, 0.7, and 0.5 and with $\sigma=0.002$ (see Fig. 1). Clearly there is a transition from a power law to an exponential decay in the presence of weak noise in each of the three cases for large n. Note, however, that the initial decay, although exponential, is at a significantly different rate and the slope of $\Gamma(n)$ settles down to the asymptotic value *gradually* after a large number of iterations.

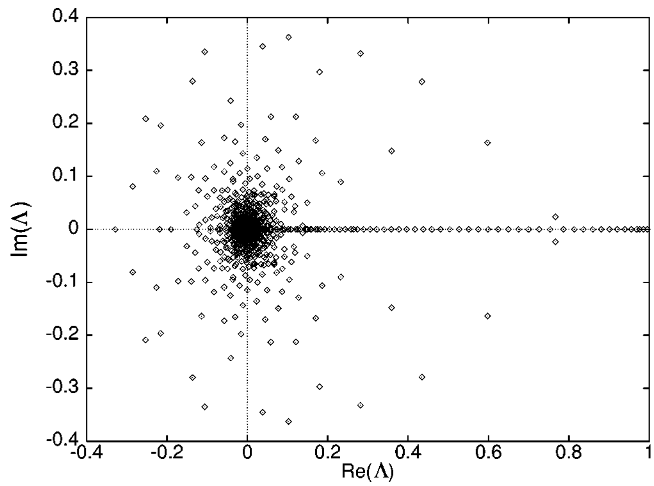

FIG. 2. The spectrum of $\mathcal{L}$ for $s=0.7$ and $\sigma=0.002$.

Two inferences can be drawn from this transition from power law to exponential behavior. First, the presence of weak noise makes the eigenvalues [of $\mathcal{L}$, see Eq. (6)] discrete. Also, there are closely spaced eigenvalues with small differences in their real parts around the leading eigenvalue Λ_0, which leads to the gradual change in slope of $\Gamma(n)$.

The discreteness of the spectrum follows from the fact that the noisy kernel is integrable and bounded. The corresponding Fredholm determinant [10] is thus entire and its zeros $(1/\Lambda_\alpha)$ are isolated. Thus, even for very weak noise, the spectrum is discrete although the transition time may be too large for the final exponential decay to be observed experimentally.

The closely spaced eigenvalues around Λ_0 are possibly remnants of the continuous spectrum that exists for the noiseless case. In order to understand this better, we have evaluated the eigenvalues of $\mathcal{L}$ by discretizing the integral equation and diagonalizing the resulting matrix [9]. Recall that a Fredholm integral equation exists as a limit of a discrete sum [10], so that a matrix representation is adequate so long as its order is large and spurious eigenvalues are eliminated. Figure 2 shows a plot of the eigenvalues of $\mathcal{L}$ for $s=0.7$ and $\sigma=0.002$ [see Eq. (9)] obtained using a matrix of size 2500×2500. We have checked that the relevant eigenvalues on the positive real axis have converged to the fourth significant digit.

Clearly, the closely spaced eigenvalues along the real line do not allow the survival fraction to be dominated by the leading eigenvalue for small n. An order of magnitude evaluation of the transition time can be made by noting that $\Gamma(n)\simeq e^{-n\ln(1/\Lambda_0)}(d_0+d_1e^{-n\ln(\Lambda_0/\Lambda_1)})$, where Λ_1 is the next to leading eigenvalue and $d_\alpha=c_\alpha\int dx\,\phi_\alpha(x)$. Thus, for $n \gg n_{trans}=1/\ln(\Lambda_0/\Lambda_1)$, the leading eigenvalue dominates. For the three values of s considered, $n_{trans}\simeq 98,105,119$ for $s=0.5,0.7,0.9$, respectively so that exponential decay sets in first for $s=0.5$ as observed in Fig. 1. Thus the difference between the leading and the next to leading eigenvalues gives a good qualitative picture and fixes a lower bound for the transition time. Note also that, for each of the three s values considered, the leading eigenvalue accurately reproduces the asymptotic decay of the survival fraction (see Table I).

We now turn our attention to the effect of noise in non chaotic systems. Specifically, we shall consider triangular billiards, which are nonchaotic though generically nonintegrable. The only integrable examples are the $(\pi/3,\pi/3,\pi/3)$, $(\pi/2,\pi/3,\pi/6)$, and $(\pi/2,\pi/4,\pi/4)$ triangles, while all other rational triangles have at least one internal angle of the form $m\pi/n,\ m>1$ and are nonintegrable. Their invariant surface, though two dimensional, is not a torus but topologically equivalent to a sphere with multiple holes. Also, these systems are nonchaotic although irrational triangles are possibly ergodic and even display the weak mixing property [11]. A linear stability analysis shows that the Jacobian matrix has unit eigenvalues and hence these billiards are marginally stable [12].

TABLE I. The leading eigenvalue Λ_0 computed by discretizing $\mathcal{L}(x,y)$ compared to the best fit value Λ_{fit} (see Fig. 1) for three different parameter values.

s	Λ_0	Λ_{fit}
0.5	0.99470	0.99467
0.7	0.99508	0.99502
0.9	0.99553	0.99546

Consider such a triangular billiard of unit perimeter and let q,p denote the Birkhoff coordinates. Here q is measured along the boundary while $p=\sin(\theta)$, where θ is the angle between the ray and the inward normal at the boundary point q. Thus $q\in[0,1]$ and $p\in[-1,1]$. In a typical experiment, one considers an initial uniform distribution of particles $(\sim10^8)$ in this phase space, evolving freely between bounces and reflecting specularly from the walls. The particles are allowed to escape through a small opening at q_0 of extent Δq_0 $(=0.005)$. For both the integrable $(\pi/2,\pi/3)$ and nonintegrable $(18\pi/31,17\pi/97)$ triangles considered, the initial decay is exponential while the asymptotic decay is a power law, $\Gamma(n)\sim n^{-\beta}$, with $\beta=1.035$ in the integrable case and $\beta=1.085$ in the nonintegrable (NI) case. Thus preexponential decays are not exceptional and can persist for a long time in marginally stable systems.

A more realistic situation should include noise, however. For instance, imperfections can give rise to maps of the type considered in Eq. (1). This leads to interesting results. For Gaussian noise (in Birkhoff momentum) with $\sigma=0.000\,001$, a single exponential decay dominates the survival fraction in the NI case for $n>5000$ while in the integrable case the transition continues beyond $n=14\,000$. Moreover, the closely spaced eigenvalues lead to a quasialgebraic decay in the interval $6000<n<14\,000$ for the integrable case. Note that most trajectories remain largely unaffected for several hundred bounces for the value of σ considered, so that $\Gamma(n)$ closely follows the noiseless case initially. Thus, even when the asymptotic decay is exponential, transition times can be very large.

In the weak noise case, the evolution operator can be approximated as

$$(\mathcal{L}\circ\phi)(x)\simeq\xi(x)+\frac{\sigma^2}{2}\xi''(x), \tag{14}$$

where $\xi(x)=\Sigma_i\phi(x)/|f'(f_i^{-1}(x))|$ and the summation is over the different branches of the inverse map. Using a polynomial basis [9], a matrix representation of the operator can be constructed where the elements $\mathcal{L}_{mn}=(d^m/dx^m)\{\mathcal{L}\circ(x^n/n!)\}$. Writing σ as $\sigma_1+\sigma_0$ where $\sigma_0\simeq 0^+$, a perturbation calculation shows that the eigenvalues (and thus the difference between Λ_0 and Λ_1) decrease as σ_1^2 when σ_1 is small. The transition time therefore decreases with noise.

For $\sigma=0.000\,05$ (see Fig. 3), the transition time decreases significantly and exponential decay sets in for $n>3500$ in the integrable case while the leading eigenvalue dominates from the beginning in the nonintegrable case. Thus the gap between Λ_0 and Λ_1 increases with noise.

In conclusion, the broad picture that emerges from these numerical experiments is as follows. (i) Additive noise makes the spectrum of the evolution operator discrete. (ii) When the dynamics is intermittent or regular and the noise weak, exponential decays may emerge only asymptotically due to the presence of closely spaced eigenvalues around the leading eigenvalue Λ_0. These are remnants of the continuous spectrum that exists in the zero noise case. The transition phase in such a situation can mimic an algebraic decay.

There are important fallouts of this conclusion. In experimental situations where noise is inevitable, signatures of exponential decays are not necessarily indicative of chaotic dynamics. For instance, semiclassical theory links Lorentzian line shapes observed in experiments on ballistic transport in chaotic microstructures to the exponential decay in the survival fraction [13]. The present analysis, however, shows that noisy intermittent dynamics can also give rise to Lorentzian line shapes, and it is interesting to note that there are instances where observations on regular or marginally stable cavities have been found to be no different from those on chaotic cavities [13].

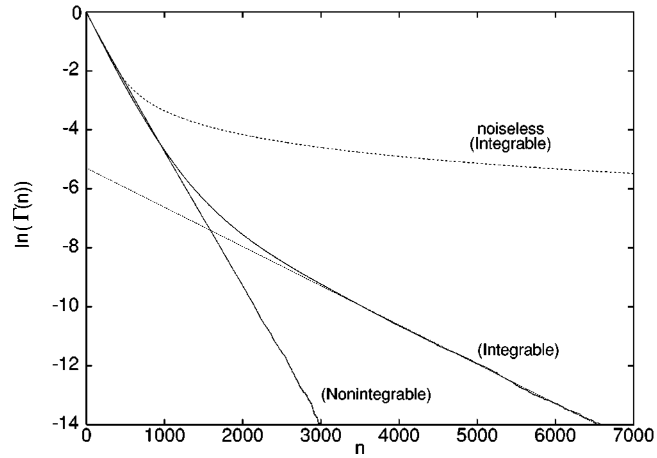

FIG. 3. $\Gamma(n)$ for the $(\pi/2,\pi/3,\pi/6)$ (integrable) and the $(18\pi/31,17\pi/97)$ (nonintegrable) triangles, along with the best fit exponential (dashed line) in the integrable case. Here $\sigma=0.000\,05$ and $\Delta q_0=0.005$.

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